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HWI

MATH 534A
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1. Tu, 1.2

A C^∞ function very flat at 0

Let $f(x)$ be

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0. \end{cases}$$

(a) Show by induction that for $x > 0$ and $k \ge 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.

Solution. Check the first derivative. By the chain rule,

$$\begin{aligned} f^{(1)}(x) &= \frac{1}{x^2}e^{-1/x} \\ &= p_2(1/x)f(x) \end{aligned}$$

where $p_2(y) = y^2$. So, $k = 1$ works. Now assume $k = m$ satisfies the given relation; that is

$$f^{(m)}(x) = p_{2m}(1/x)e^{-1/x}.$$

The $(m+1)$st derivative then is

$$\begin{aligned} f^{(m+1)}(x) &= -\frac{1}{x^2}p'_{2m}(1/x)e^{-1/x} + \frac{1}{x^2}p_{2m}(1/x)e^{-1/x} \\ &= e^{-1/x}\left(-q_{2m+1}(1/x) + q_{2_m+2}(1/x)\right) \\ &= p_{2(m+1)}(1/x)e^{-1/x}. \end{aligned}$$

So, the formula holds. □

(b) Prove that f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \ge 0$.

Proof. First, look at $f(x)$. The limit as $x \to 0$ is 0, as $y = \frac{1}{x}$ goes to ∞ as $x \to 0$ and $e^{-\infty} = 0$.

f is C^∞ from part (a). Consider an inductive argument, where $f^{(k)} = 0$, so the following limit exists

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} \\ &= \lim_{x\to 0^+} \frac{f^{(k)}(x)}{x}. \end{aligned}$$

The limit as $x \to 0$ from the left is 0 from the definition of f. Now, the limit from the right is,

$$\lim_{x \to 0^+} \frac{f^{(k)}(x)}{x} = \lim_{x \to 0^+} \frac{p_{2k}(1/x)e^{-1/x}}{x}.$$

Pick a convenient change of variables $y \mapsto \frac{1}{x}$ and the limit becomes

$$\begin{aligned} \lim_{x \to 0^+} \frac{p_{2k}(1/x)e^{-1/x}}{x} &= \lim_{y \to \infty} \frac{p_{2k+1}(y)}{e^y} \\ &= \frac{\infty}{\infty}. \end{aligned}$$

Apply L' Hospítal's rule $2k+1$ times and the limit is

$$\begin{aligned} \lim_{y \to \infty} \frac{p_0(y)}{e^y} &= \frac{c}{\infty}, \ c \in \mathbb{R} \\ &= 0. \end{aligned}$$

Therefore, the limits from the right and left agree and

$$f^{(k+1)}(0) = 0.$$

The argument follows inductively for all $m > k+1$.

$\therefore$ f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$. □

2. Tu, 1.3

A diffeomorphism of an open interval with $\mathbb{R}$

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open subsets. A C^∞ map $F : U \to V$ is called a *diffeomorphism* if it is bijective and has C^∞ inverse $F^{-1} : V \to U$.

(a) Show that the function $f : (-\pi/2, \pi/2) \to \mathbb{R}$, $f(x) = \tan x$, is a diffeomorphism.

Solution. First, a useful theorem;

Theorem 1. *Given sets U, V. A function $f : U \to V$ is bijective iff $f^{-1} : V \to U$ is a function.*

For the problem given, $f^{-1}(x) = \arctan(x)$ from $\mathbb{R} \to (-\pi/2, \pi/2)$ is a function because it passes the vertical line test for functions; see Figure 1.

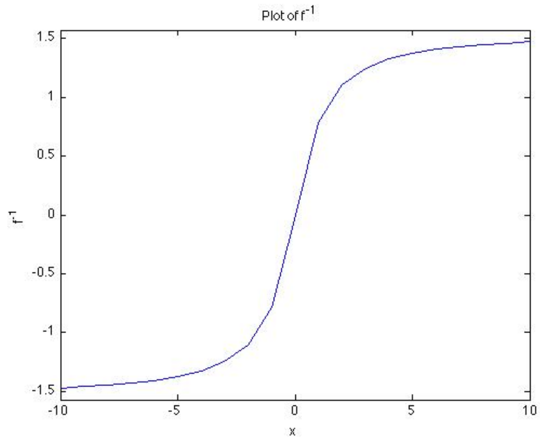

Figure 1: Plot of $\arctan(x)$

Therefore, $f : (-\pi/2, \pi/2) \to \mathbb{R}$ is bijective.

Consider $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$. For $x \in \mathbb{R}$, this function and all of its derivatives are well-defined. Therefore, $f^{-1} \in C^\infty$.

Therefore, f is a diffeomorphism. □

(b) Find a linear function $h : (a, b) \to (-1, 1)$, thus proving that any two finite open intervals are diffeomorphic.

Solution. Consider the function

$$h(x) = \frac{1}{(b-a)}(x-a) - \frac{1}{2}$$

which is a linear mapping from $(a, b) \to (-1, 1)$. □

The composite $f \circ h : (a, b) \to \mathbb{R}$ is then a diffeomorphism of an open interval to $\mathbb{R}$.

3. Tu, 1.4

A diffeomorphism of an open ball with $\mathbb{R}^n$

(a) Show that the function $h : (-\pi/2, \pi/2) \to [0, \infty)$,

$$h(x) = \begin{cases} e^{-1/x} \sec x & \text{for } x \in (0, \pi/2) \\ 0 & \text{for } x \leq 0, \end{cases}$$

is C^∞ on $(-\pi/2, \pi/2)$, strictly increasing on $[0, \pi/2)$, and satisfies $h^{(k)} = 0$ for all $k \geq 0$. (*Hint*: Let $f(x)$ be the function of Example 1.3 and let $g(x) = \sec x$. Then $h(x) = f(x)g(x)$. Use the properties of $f(x)$.)

Solution. Define functions f, g accordingly

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$
$$g(x) = \begin{cases} \sec(x) & \text{for } x \in (-\pi/2, \pi/2) \\ 0 & \text{otherwise} \end{cases}$$

We have proven that $f \in C^\infty$. In $(-\pi/2, \pi/2)$, g is differentiable since $1/\cos(x)$ is non-zero for all such x in this interval. It follows that $g \in C^\infty$. The product of C^∞ functions is C^∞, therefore $h = fg \in C^\infty$.

f is strictly increasing on $[0, \pi/2)$, as is g. Therefore, the product $h = fg$ is also strictly increasing on $[0, \pi/2)$.

Now, for the derivative. Using the product rule, we have the relation

$$h^{(k)}(0) = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(0) g^{(k-i)}(0) = 0.$$

because, from problem 1.2, we know that all derivatives of f at 0 are 0. □

(b) Define the map $F : B(0, \pi/2) \subset \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x) = \begin{cases} h(|x|)\frac{x}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Show that $F : B(0, \pi/2) \to \mathbb{R}^n$ is a diffeomorphism.

Solution. Use the fact that h is strictly increasing on $(0, \pi/2) \subset \mathbb{R}$, and it is clear that h^{-1} is a function. Therefore, F^{-1} is a function, and the mapping F is bijective where

$$F^{-1}(x) = \begin{cases} h^{-1}(|x|)\frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Now, we need that F^{-1} is C^∞. To accomplish this, we need only be concerned about the differentiability of h^{-1}. Finding the inverse analytically would be difficult, so I am going to try a geometric approach. Figure 2 shows the plot of h^{-1}.

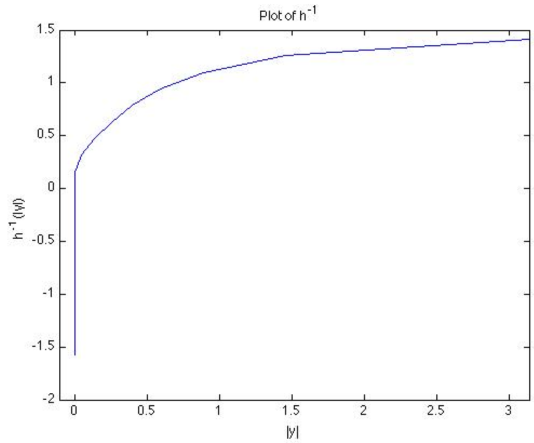

Figure 2: Plot of $h^{-1}(|y|)$

If the plot $x-$axis were allowed to extend past π, we would see that the function is smooth on all of $\mathbb{R}^+ \setminus \{0\}$; that is, $h^{-1}(|x|)$ is everywhere differentiable, so $h^{-1} \in C^\infty$. As 0 is not included in the pullback of f, we need not worry about the infinite slope at 0.

Therefore, F^{-1} is a diffeomorphism. $\square$

4. Tu, 1.7

Bijective C^∞ maps

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Show that f is a bijective C^∞ map, but that f^{-1} is not C^∞. (In complex analysis a bijective holomorphic map $f : \mathbb{C} \to \mathbb{C}$ necessarily has a holomorphic inverse.)

Solution. The map $f^{-1}(x) = x^{1/3}$ is a function from $\mathbb{R} \to \mathbb{R}$, therefore f is bijective. However, take one derivative

$$\frac{d}{dx} f^{-1}(x) = -\frac{1}{x^{2/3}}$$

which does not exist at $x = 0$. So, $f^{-1} \notin C^\infty$.

Therefore f^{-1} is not a diffeomorphism. $\square$

5. Tu, 2.1

Vector fields

Let X be the vector field $x\partial/\partial x + y\partial/\partial y$ and $f(x, y, z)$ the function $x^2 + y^2 + z^2$ on $\mathbb{R}^3$. Compute Xf.

Solution.

$$\begin{aligned} Xf &= \big(x\partial/\partial x + y\partial/\partial y\big)(x^2 + y^2 + z^2) \\ &= 2x^2 + 2y^2. \end{aligned}$$

The directionality should be inherited from how the partials are taken, i.e.

$$Xf = 2x^2\hat{x} + 2y^2\hat{y}$$

□

6. Tu, 2.2

Algebra structure on C_p^∞

Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.

Solution. Consider $f, g \in C_p^\infty$. This means that there are open sets $U, V \ni p$ on which f, g are defined. The intersection of these sets $(U \cap V)$ will be where the functions f, g agree. So, we can define $f+g$ on $U \cap V$ and return a germ at p; that is, $f+g \in C_p^\infty$ on the set $U \cap V$. The same argument follows for fg and cf, where $c \in \mathbb{R}$. The obvious exception is that cf will be a germ on the set U, on which f was originally defined.

To prove that addition is commutative is not difficult. Taking into account the fact that f agrees with g on $U \cap V$, we have (for $x \in U \cap V$)

$$\begin{aligned} f(x) + g(x) &= 2f(x) \\ &= 2g(x) \\ &= g(x) + f(x). \end{aligned}$$

□

7. Tu, 2.3

Vector space structure on derivations at a point

Let D and D' be derivations at p in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that

(a) the sum $D + D'$ is a derivation at p.

Proof. We need to show that $D + D'$ is (1) linear and (2) obeys the Leibniz rule. Assume $f, g \in C_p^\infty$. $r \in \mathbb{R}$ and D, D' are derivations.

(1) is easy.

$$\begin{aligned} (D+D')(f+g) &= Df + Dg + D'f + D'g \\ &= (D+D')f + (D+D')g \\ (D+D')(rf) &= D(rf) + D'(rf) \\ &= rD(f) + rD'(f) \\ &= r(D+D')(f) \end{aligned}$$

For (2), D and D' both satisfy the Leibniz rule, therefore

$$\begin{aligned} (D+D')(fg) &= D(fg) + D'(fg) \\ &= D(f)g + fD(g) + D'(f)g + fD'(g) \\ &= g(D+D')(f) + f(D+D')(g) \end{aligned}$$

Therefore, $D + D'$ is a derivation. □

(b) the scalar multiple cD is a derivation at p.

Proof. We need to show that cD is (1) linear and (2) obeys the Leibniz rule. Assume $f, g \in C_p^\infty$. $r \in \mathbb{R}$ and D is a derivation.

Again, (1) is easy

$$\begin{aligned} cD(f+g) &= c(D(f)+D(g)) \\ &= cD(f)+cD(g) \\ cD(rf) &= crD(f) \\ &= rcD(f) \end{aligned}$$

so cD is linear.

For (2),

$$\begin{aligned} cD(fg) &= c(D(f)g+fD(g)) \\ &= cD(f)g+cD(g)f \end{aligned}$$

so cD satisfies the Leibniz rule.

Therefore, cD is a derivation. $\square$

8. Tu, 3.9

Let $\alpha^1, \ldots, \alpha^k$ be 1-covectors on a vector space V. Show that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \ldots, \alpha^k$ are linearly independent in the dual space V^*.

Solution. ($\Longrightarrow$) It is easiest to show the contrapositive. Assume that the α^k are linearly dependent, then WLOG assume that the α^l is a combination of the other guys

$$\alpha^l = \sum_{j \neq l}^{k} c_j \alpha^j$$

Then the wedge product becomes

$$\begin{aligned} \alpha^1 \wedge \cdots \wedge \alpha^l \wedge \cdots \wedge \alpha^k &= \alpha^1 \wedge \cdots \wedge \Big(\sum_{j \neq l}^{k} c_j \alpha^j\Big) \wedge \cdots \wedge \alpha^k \\ &= \sum_{j \neq l}^{k} c_j (\alpha^1 \wedge \cdots \wedge \alpha^j \wedge \cdots \wedge \alpha^k) \\ &= \sum_{j \neq l}^{k} c_j \cdot 0 \\ &= 0 \end{aligned}$$

Therefore, if $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$, then $\alpha^1, \ldots, \alpha^k$ are linearly independent in the dual space V^*.

($\Longleftarrow$) Suppose the α^k are linearly independent. Therefore, they can be extended to a basis for the dual space V^*. Call this basis $v^1, \ldots, v^k$. Evaluating the wedge product of these linear functionals at this basis gives

$$\begin{aligned}(\alpha^1 \wedge \cdots \wedge \alpha^k)(v^1, \ldots, v^k) &= \det(\alpha^j(v_j)) \text{ by Prop. 3.28} \\ &= \det(\delta^i_j) \\ &= 1\end{aligned}$$

which is non-zero. Therefore, if $\alpha^1, \ldots, \alpha^k$ are linearly independent in the dual space V^*, then $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$.

Hence, if $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \ldots, \alpha^k$ are linearly independent in the dual space V^*. □

9. Tu, 3.10

Let α be a nonzero 1-covector and ω a k-covector on a finite-dimensional vector space V. Show that $\alpha \wedge \omega = 0$ if and only if $\omega = \alpha \wedge \tau$ for some $(k-1)$-covector τ on V.

Solution. This problem uses the book's convention for J notation.

The reverse direction is easiest, so consider it first.

($\Longrightarrow$) Assume $\omega = \alpha \wedge \tau$. Then,

$$\begin{aligned}\alpha \wedge \omega &= \alpha \wedge (\alpha \wedge \tau) \\ &= (\alpha \wedge \alpha) \wedge \tau \text{ by associativity} \\ &= 0\end{aligned}$$

($\Longleftarrow$) Suppose, $\alpha \wedge \tau = 0$. Extend α to a basis $\alpha^1, \ldots, \alpha^n$ for some n, where $\alpha = \alpha^1$. We can represent ω as a linear combination of elements in this basis; that is, $\sum_J c_J \alpha^J$. Now, we have

$$\sum_J c_j \alpha^j = \sum_{j_1 \neq 1} c_j \alpha^j$$

and

$$\begin{aligned}\alpha \wedge \omega &= \sum_{j_1 \neq 1} cJj\alpha \wedge \alpha^J \\ &= 0\end{aligned}$$

The set $\{\alpha \wedge \alpha^J\}_{j_1 \neq 1}$ forms a subset of a basis for $A_{k+1}(V)$, so all entries of the set must be linearly independent. Therefore, $c_J = 0$ for all $j_1 \neq 1$. Therefore,

$$\begin{aligned}\omega &= \sum_{j_1 = 1} c_J \alpha^J \\ &= \alpha \wedge \sum_{j_1 = 1} c_J (\alpha^{j_2} \wedge \cdots \wedge \alpha^{j_k})\end{aligned}$$

which is a $(k-1)$-covector. This completes the proof. □

10. Tu, 3.11

For any linear map $L : V \to W$ of vector spaces and any positive integer k, there is a *pullback map* $L^* : A_k(W) \to A_k(V)$ defined by

$$L^*(f)(v_1, \dots, v_k) = f(L(v_1), \dots, L(v_k))$$

for all $v_1, \dots, v_k \in V$. Show that if $L : V \to V$ is a linear operator of a vector space V of dimension n, then $L^* : A_n(V) \to A_n(V)$ is multiplication by the determinant of L.

Solution. Let $e_1, \dots, e_n$ be a basis for V and $\alpha^1, \cdots, \alpha^k$ be the dual basis for V^*. Then a basis for $A_n(V)$ is $\alpha^1 \wedge \cdots \wedge \alpha^n$ and $L^*(\alpha^1 \wedge \cdots \wedge \alpha^n) = c\alpha^1 \wedge \cdots \wedge \alpha^n$ for some constant c.

Suppose $L(v_j) = \sum_i a^i_j e_i$, then

$$\begin{aligned}
(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \dots, v_n) &= (\alpha^1 \wedge \cdots \wedge \alpha^k)(\sum_i a^i_1 e_i, \dots, \sum_i a^i_n e_i) \\
&= \det(\alpha^i(\sum_i a^i_j e_i) \\
&= \det(\sum_j L(v_j)) \\
&= \sum_j \det(L(v_j)) \\
&= \det(L)
\end{aligned}$$

which implies that c from above must be $\det(L)$. □

11. Tu, 4.1

Let ω be the 1-form $zdx - dz$ and X be the vector field $y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $d\omega$.

Solution.

$$\begin{aligned}
\omega(X) &= zy + 0 * x - 1 * 0 \\
&= yz \\
d\omega &= \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j} dx^j \right) \wedge dx^I \\
&= 0 + 0 + dx \wedge dz \\
&= dx \wedge dz
\end{aligned}$$

□

12. Tu, 4.6

To a 1-covector $\alpha = a_1 dx + a_2 dy + a_3 dz$ on $\mathbb{R}^3$ we associate the vector $\mathbf{v}_\alpha = (a_1, a_2, a_3)$ in $\mathbb{R}^3$; to a 2-covector $\gamma = c_1 dy \wedge dz + c_2 dz \wedge dx + c_3 dx \wedge dy$ on $\mathbb{R}^3$, we associate the vector $\mathbf{v}_\gamma = (c_1, c_2, c_3)$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$; if $\alpha = a_1 dx + a_2 dy + a_3 dz$ and $\beta = b_1 dx + b_2 dy + b_3 dz$, then $\mathbf{v}_{\alpha \wedge \beta} = \mathbf{v}_\alpha + \mathbf{v}_\beta$.

Solution. Calculate the wedge product $\alpha \wedge \beta$,

$$\begin{aligned}
\alpha \wedge \beta &= (a_1 dx + a_2 dy + a_3 dz) \wedge (b_1 dx + b_2 dy + b_3 dz) \\
&= a_1 b_1 dx \wedge dx + a_1 b_2 dx \wedge dy + a_1 b_3 dx \wedge dz \\
&+ a_2 b_1 dy \wedge dx + a_2 b_2 dy \wedge dy + a_2 b_3 dy \wedge dz \\
&+ a_3 b_1 dz \wedge dx + a_3 b_2 dz \wedge dy + a_2 b_3 dz \wedge dz \\
&= a_1 b_1 (0) + a_1 b_2 dx \wedge dy + a_1 b_3 (-1) dz \wedge dx \\
&+ a_2 b_1 (-1) dx \wedge dy + a_2 b_2 (0) + a_2 b_3 dy \wedge dz \\
&+ a_3 b_1 dz \wedge dx + a_3 b_2 (-1) dy \wedge dz + a_2 b_3 (0) \\
&= (a_2 b_3 - a_3 b_2) dy \wedge dz + (a_3 b_1 - a_1 b_3) dz \wedge dx + (a_1 b_2 - a_2 b_1) dx \wedge dy,
\end{aligned}$$

to which we associate the vector $(a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$. This is the cross product of the vectors associated with α and β. □

13. Tu, 4.7

If ω is a k-covector on a vector space V and $v \in V$, the *interior multiplication* or *contraction* of ω with v is the $(k-1)$-covector $\iota_v \omega$ defined by

$$(\iota_v \omega)(v_2, \ldots, v_k) = \omega(v, v_2, \ldots, v_k)$$

for all $v_2, \ldots, v_k \in V$. If $\alpha^1, \ldots, \alpha^k$ are 1-covectors on V, prove that

$$\iota_v (\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i+1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k,$$

where the caret $\widehat{\ }$ over α^i means that α^i is omitted from the wedge product. (*Hint*: Use the determinant formula for the wedge product of 1-covectors (Proposition 3.28).)

Proof.

$$\begin{aligned}
\iota_v (\alpha^1 \wedge \cdots \wedge \alpha^k)(v_2, \ldots, v_k) &= \alpha^1 \wedge \cdots \wedge \alpha^k (v, v_2, \ldots, v_k) \\
&= \det \begin{bmatrix} \alpha^1(v) & \alpha^1(v_2) & \ldots & \alpha^1(v_k) \\ \alpha^2(v) & \alpha^2(v_2) & \ldots & \alpha^2(v_k) \\ \vdots & \vdots & & \vdots \\ \alpha^k(v) & \alpha^k(v_2) & \ldots & \alpha^k(v_k) \end{bmatrix} \\
&= \sum_{i=1}^{k} (-1)^{i+1} \alpha^i(v) \det(\alpha^l(v_j)), \ 1 \le l \le k, l \ne 1 \text{ and } 2 \le j \le k \\
&= \sum_{i=1}^{k} (-1)^{i+1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k (v_2, \ldots, v_k)
\end{aligned}$$

where I used Prop. 3.28 and expansion along the first column for evaluation of the determinant. □